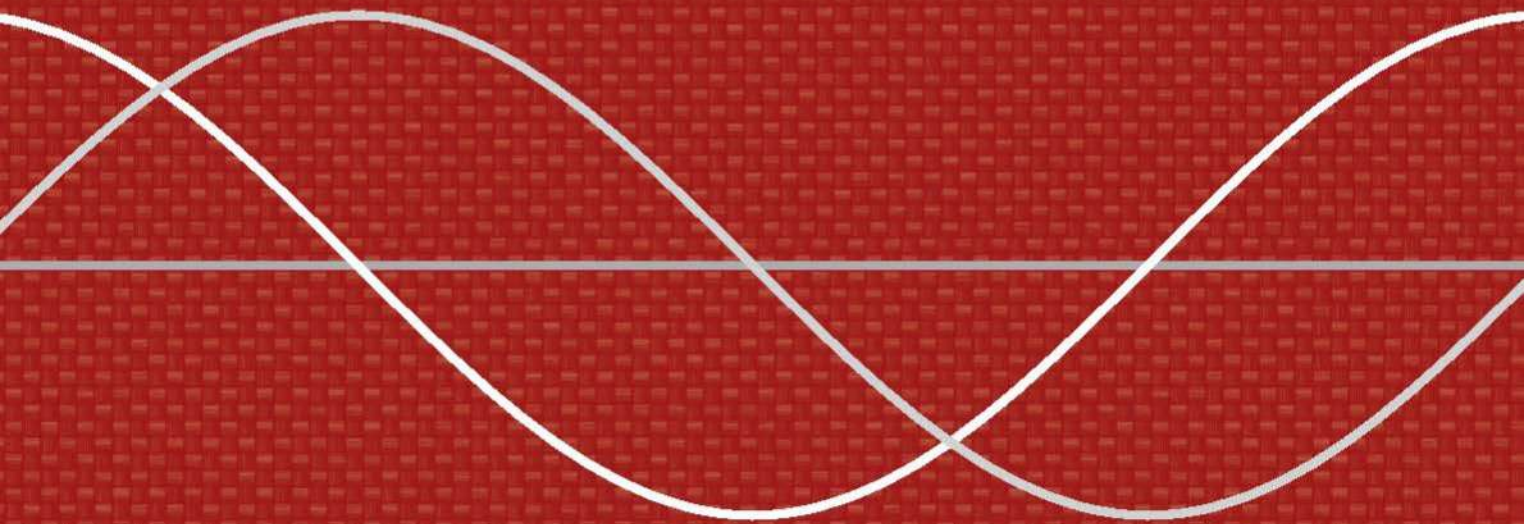


CALCULUS



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CALCULUS

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To Leibniz and Newton

—*Laura Taalman*

To Newton and Leibniz

—*Peter Kohn*

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ABOUT THE AUTHORS

Laura Taalman and Peter Kohn are professors of mathematics at James Madison University, where they have taught calculus for a combined total of over 30 years.

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Calculus books have become full of clutter, distracting margin notes, and unneeded features. This calculus book clears out that clutter so that students can focus on the important ideas of calculus. Our goal was to create a clean, streamlined calculus book that is accessible and readable for students while still upholding the standards required in science, mathematics, and engineering programs, and that is flexible enough to accommodate different teaching and learning styles.

Linear Flow with Clean Margins

One thing that is distinctive about this calculus book is that it follows a linear writing style. Figures and equations flow with the text as part of a clear, structured exposition instead of being scattered about in the margins. We feel that this approach greatly increases the clarity of the book and encourages focused reading.

Exposition Before Calculation

Another distinctive feature of this book is that in each section we have separated the exposition and illustrative examples from the longer, more complicated calculational examples. Including these longer examples separately from the exposition increases flexibility: Students who want to read and understand the development of the material can do so without being bogged down or distracted by large examples, while students who want to use the book as a reference for looking up examples that are similar to homework problems can also do that.

Examples to Learn From

Within the exposition of each section are short examples that quickly illustrate the concepts being developed. Following the exposition is a set of detailed, in-depth examples that explore both calculations and concepts. We took great pains to provide many steps and illustrations in each example in order to aid the student, including details about how to get started on a problem and choose an appropriate solution method. One of the elements of the book that we are most proud of is the “Checking the Answer” feature, which we have included after selected examples to encourage students to learn how to check their own answers.

Building Mathematics

We were very careful in this book to approach mathematics as a discipline that is developed logically, theorem by theorem. Whenever possible, theorems are followed by proofs that are written to be understood by students. We have included these proofs because they are part of the logical development of the material, but we have clearly labeled and indented each proof to indicate that it can be covered or skipped, according to instructor preference. Each exercise set contains an optional subsection of proofs, many of which are accessible even to beginning students. In addition, we have emphasized the interconnections among topics by providing “Thinking Back” and “Thinking Forward” exercises in each section and “Capstone” problems at the end of each chapter.

Consistency and Reliability

Another improvement in this book is that it has a consistent and predictable structure. For example, instructors can rely on every section concluding with a “Test Your Understanding”

feature which includes five questions that students can use to self-test and that instructors can choose to use as pre-class questions. The exercises are always consistently split into subsections of different types of problems: “Thinking Back,” “Concepts,” “Skills,” “Applications,” “Proofs,” and “Thinking Forward.” In addition, the “Concepts” subsection always begins with a summary exercise, eight true/false questions, and three example construction exercises. Instructors and students alike can rely on this consistent structure when assigning exercises and choosing a path of study.

Flexibility

We recognize that instructors use calculus books in many different ways and that the real direction of a calculus course comes from the instructor, not any book. The streamlined, consistent structure of this book makes it easy to use with a wide variety of courses and pedagogical styles. In particular, instructors will find it easy to include or omit sections, proofs, examples, and exercises consistently according to their preferences and course requirements. Students can focus on mathematical development or on examples and calculations as they need to throughout the course. Later, they can use the book as a reliable reference.

We think it will be immediately clear to anyone opening this book that what we have written is substantially different from the other calculus books on the market today while still following the standard topics taught in most modern science, mathematics, and engineering calculus courses. Our hope is that faculty who use the book will find it flexible for different pedagogical approaches and that students will be able to read it on different levels as they learn to understand the beauty of calculus.

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Do some of your calculus students struggle with algebra and precalculus material? The Taalman/Kohn *Calculus* series has a ready-made option for such students, called *Calculus I with Integrated Precalculus*. This option includes all the material in Chapters 0–6 of Taalman/Kohn *Calculus*, but in a different order and with supplementary precalculus and algebra material.



- ▶ Chapters 0–3 of *Calculus I with Integrated Precalculus* cover the same development of differential calculus topics as Chapters 0–3 in Taalman/Kohn *Calculus*, but the more complicated calculational examples are deferred to later chapters.
- ▶ Chapters 4–6 of *Calculus I with Integrated Precalculus* revisit differential calculus through the lens of studying progressively more challenging types of functions. Any exercises or examples from Taalman/Kohn *Calculus* that were left out of Chapters 0–3 of *Calculus I with Integrated Precalculus* are included in Chapters 4–6. The requisite background precalculus and algebra material is built from the ground up.
- ▶ Chapters 7–9 of *Calculus I with Integrated Precalculus* are identical to Chapters 4–6 of Taalman/Kohn *Calculus* and cover all topics from integral calculus.

Students who learn Calculus I from *Calculus I with Integrated Precalculus* can continue with Calculus II using Taalman/Kohn *Calculus* or any other calculus textbook. Students who have weak algebra and precalculus skills can succeed in STEM-level calculus if given the right help along the way, and *Calculus I with Integrated Precalculus* is written specifically to address the needs of those students.

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Each section opens with a **list of the three main section topics**. The list provides a focus and highlights key concepts.

3.3 THE SECOND DERIVATIVE AND CURVE SKETCHING

- ▶ Using first and second derivatives to define and detect concavity
- ▶ The behavior of the first and second derivatives at inflection points
- ▶ Using the second-derivative test to determine whether critical points are maxima, minima, or neither

Definitions are clearly boxed, numbered, and labeled for easy reference. To reinforce their importance and meaning, definitions are followed by brief, often illustrated, examples.

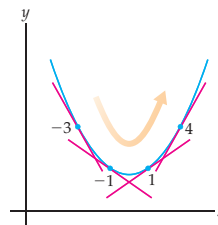
DEFINITION 3.9 Formally Defining Concavity

Suppose f and f' are both differentiable on an interval I .

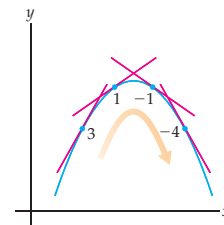
- (a) f is **concave up** on I if f' is increasing on I .
- (b) f is **concave down** on I if f' is decreasing on I .

How does this formal definition of concavity correspond with our intuitive notion of concavity? Consider the functions graphed next. On each graph four slopes are illustrated and estimated. Notice that when f is concave up, its slopes increase from left to right, and when f is concave down, its slopes decrease from left to right.

Slopes increase when f is concave up



Slopes decrease when f is concave down



Theorems are developed intuitively before they are stated formally, and simple examples inform the discussion. **Proofs** follow most theorems, although they are optional, given instructor preference.

THEOREM 3.4 Rolle's Theorem

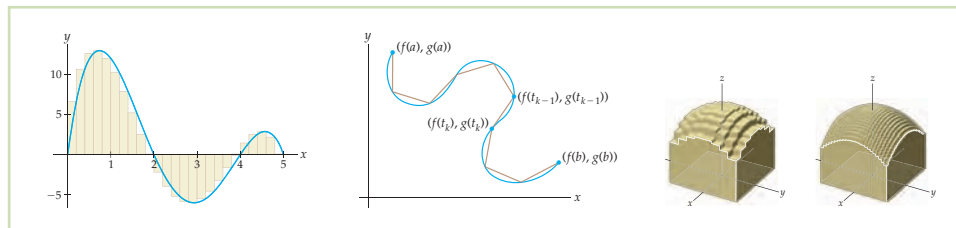
If f is continuous on $[a, b]$ and differentiable on (a, b) , and if $f(a) = f(b) = 0$, then there exists at least one value $c \in (a, b)$ for which $f'(c) = 0$.

Actually, Rolle's Theorem also holds in the more general case where $f(a)$ and $f(b)$ are equal to each other (not necessarily both zero). For example, Rolle's Theorem is also true if $f(a) = f(b) = 5$, or if $f(a) = f(b) = -3$, and so on, because vertically shifting a function by adding a constant term does not change its derivative. However, the classic way to state Rolle's Theorem is with $f(a)$ and $f(b)$ both equal to zero.

Proof. Rolle's Theorem is an immediate consequence of the Extreme Value Theorem from Section 1.4 and the fact that every extremum is a critical point. Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , with $f(a) = f(b) = 0$. By the Extreme Value Theorem, we know that f attains both a maximum and a minimum value on $[a, b]$. If one of these extreme values occurs at a point $x = c$ in the interior (a, b) of the interval, then $x = c$ is a local extremum of f . By the previous theorem, this means that $x = c$ is a critical point of f . Since f is assumed to be differentiable at $x = c$, it follows that $f'(c) = 0$ and we are done.

It remains to consider the special case where all of the maximum and minimum values of f on $[a, b]$ occur at the endpoints of the interval (i.e., at $x = a$ or at $x = b$). In this case, since $f(a) = f(b) = 0$, the maximum and minimum values of $f(x)$ must both equal zero. For all x in $[a, b]$ we would have $0 \leq f(x) \leq 0$, which means that f would have to be the constant function $f(x) = 0$ on $[a, b]$. Since the derivative of a constant function is always zero, in this special case we have $f'(x) = 0$ for all values of c in (a, b) , and we are done. ■

Color is used consistently and pedagogically in **graphs and figures** to relate like concepts. For instance, the color used for rectangles in Riemann sum approximations is also quite purposefully used for linear approximations of arc length and rectangular solid approximations of volume.



Cautions are appropriately placed at points in the exposition where students typically have questions about the nuances of mathematical thinking, processes, and notation.



CAUTION

It is important to note that although we use the notation x^{-1} to denote the reciprocal $\frac{1}{x}$, the notation f^{-1} does *not* stand for the reciprocal $\frac{1}{f}$ of f . The notation f^{-1} used in Definition 0.10 is pronounced “ f inverse.” We are now using the same notation for two very different things, but it should be clear from the context which one we mean.

Every section includes short illustrative examples as part of the discussion and development of the material. Once the groundwork has been laid, more complex **examples** and calculations are provided. Students find this approach easier to handle because the difficult calculations do not interfere with the development of why things work. Example **solutions** are explained in detail and include all the steps necessary for student comprehension.

EXAMPLE 4

Using critical points and Rolle’s Theorem to find local extrema

The function $f(x) = x(x - 1)(x - 3)$ is a cubic polynomial with one local maximum and one local minimum. Use Rolle’s Theorem to identify intervals on which these extrema exist. Then use derivatives to find the exact locations of these extrema.

SOLUTION

The roots of $f(x) = x(x - 1)(x - 3)$ are $x = 0$, $x = 1$, and $x = 3$. Since f is a polynomial, it is continuous and differentiable everywhere. Therefore Rolle’s Theorem applies on the intervals $[0, 1]$ and $[1, 3]$, and it tells us that at least one critical point must exist inside each of these intervals.

The critical points of f are the possible locations of the local extrema that we seek. To find the critical points we must solve the equation $f'(x) = 0$. It is simpler to do some algebra before differentiating:

$$f'(x) = \frac{d}{dx}(x(x - 1)(x - 3)) = \frac{d}{dx}(x^3 - 4x^2 + 3x) = 3x^2 - 8x + 3.$$

By the quadratic formula, we have $f'(x) = 0$ at the points

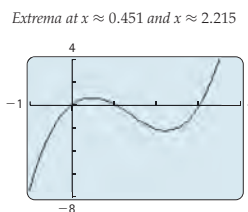
$$x = \frac{-(-8) \pm \sqrt{8^2 - 4(3)(3)}}{2(3)} = \frac{8 \pm \sqrt{28}}{6} = \frac{4 \pm \sqrt{7}}{3}.$$

These x -values are approximately $x \approx 0.451$ and $x \approx 2.215$. If we look at the graph of f , then we can see that the smaller of these two x -values is the location of the local maximum and the larger is the location of the local minimum; see the figure that follows. \square

Following many example solutions, **Checking the Answer** encourages students to learn to check their work, using technology such as a graphing calculator when appropriate.

✓ CHECKING
THE ANSWER

The graph of $f(x) = x(x - 1)(x - 3)$ is shown next. Notice that the local extrema do seem to occur at the values we just found.



Each section closes with five **Test Your Understanding** questions that test students on the concepts and reading presented in the section. Because answers are not provided, instructors may choose to use these questions for discussion or assessment.

? TEST YOUR
UNDERSTANDING

- ▶ Why could we not give a precise mathematical definition of concavity before this section?
- ▶ The domain points $x = c$ where $f''(c) = 0$ or where $f''(c)$ does not exist are the critical points of the function f' . Why?
- ▶ Why is it not clear to say a sentence such as “Because it is positive, it is concave up”? How could this information be conveyed more precisely?
- ▶ Why does it make sense that f' is increasing when f'' is positive?
- ▶ Suppose $x = c$ is a critical point with $f'(c) = 0$. Why does it make graphical sense that f has a local minimum at $x = c$ when f is concave up in a neighborhood around $x = c$?

Section Exercises are provided in a consistent format that offers the same types of exercises within each section. This approach allows instructors to tailor assignments to their course, goals, and student audience.

Thinking Back exercises ask students to review relevant concepts from previous sections and lessons.

Concepts exercises are consistently formatted to start with the following three problems:

- Problem 0 tests understanding.
- Problem 1 consists of eight true/false questions.
- Problem 2 asks the student to create examples based on their understanding of the reading.

Skills exercises offer ample practice, grouped into varying degrees of difficulty.

Applications exercises contain at least two in-depth real-world problems.

Proofs exercises can be completed by students in non-theoretical courses. Hints are often provided, and many exercises mimic work presented in the reading and examples. Often, these exercises are a continuation of a proof offered as a road map in the narrative.

Thinking Forward exercises plant seeds of concepts to come. In conjunction with the Thinking Back exercises, they offer a “tie together” of both past and future topics, thereby providing a seamless flow of concepts.

Chapter Review, Self-Test, and Capstones, found at the end of each chapter, present the following categories:

Definitions exercises prompt students to recall definitions and give an illustrative example.

Theorems exercises ask students to complete fill-in-the-blank theorem statements.

Formulas, Notation, and/or Rules exercises vary according to chapter content and ask students to show a working understanding of important formulas, equations, notation, and rules.

Skill Certification exercises provide practice with basic computations from the chapter.

Capstone Problems pull together the essential ideas of the chapter in more challenging mathematical and application problems.

ACKNOWLEDGMENTS

There are many people whose contributions to this project have made it immeasurably better. We are grateful to the many instructors from across the United States and Canada who have offered comments that assisted in the development of this book:

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Finally, we would like to thank our families and friends for putting up with us during the years of stress, turmoil, and tedium that inevitably come with any book project. Without their support, this book would not have been possible.

Learning something new can be both exciting and daunting. To gain a full understanding of the material in this text, you will have to read, you will have to think about the connections between the new topics and the topics that were previously presented, and you will have to work problems—many, many problems.

The structure of this text should help you understand the material. The material is laid out in a linear fashion that we think will facilitate your understanding. Each section is separated into two main parts: first, a presentation of new material and then second, a set of *Examples and Explorations*, where you will find problems that are carefully worked through. Working through these examples on your own, as you read the steps for guidance, will help prepare you for the exercises.

Reading a mathematics book isn't like reading a novel: You may have to read some parts more than once, and you may need to make notes or work things out on paper. Pay special attention to the "Checking Your Answer" features, so that you can learn how to check your own answers to many types of questions.

To succeed in calculus, you need to do homework exercises. The exercises in every section of this text are broken into six categories: "Thinking Back," "Concepts," "Skills," "Applications," "Proofs," and "Thinking Forward."

- As the title suggests, the *Thinking Back* problems are intended to tie the current material to material you've seen in previous sections or even previous courses.
- The *Concepts* problems are designed to help you understand the main ideas presented in the section without a lot of calculation. Every group of *Concepts* exercises begins by asking you to summarize the section, continues with eight true/false questions, and then asks for three examples illustrating ideas from the section.
- The bulk of the exercises in each section consists of *Skills* problems that may require more calculation.
- The *Applications* exercises use the concepts from the section in "real-world" problems.
- The *Proofs* exercises ask you to prove some basic theory from the section.
- Finally, the *Thinking Forward* questions use current ideas to introduce topics that you will see in subsequent sections.

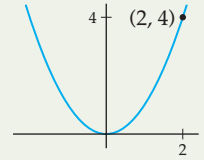
We hope this structure allows you to tie together the material as you work through the book. We have supplied the answers to the odd-numbered exercises, but don't restrict yourself to those problems. You can check answers to even-numbered questions by hand or by using a calculator or an online tool such as wolframalpha.com. After all, on a quiz or test you won't have the answers, so you'll have to know how to decide for yourself whether or not your answers are reasonable.

Some students may like to work through each section "backwards," starting by attempting the exercises, then checking back to the examples as needed when they get stuck, and, finally, using the exposition as a reference when they want to see the big picture. That is fine; although we recommend that you at least try reading through the sections in order to see how things work for you. Either way, we hope that the separation of examples from exposition and the division of homework problems into subsections will help make the process of learning this beautiful subject easier. We have written this text with you, the student, in mind. We hope you enjoy using it!

Functions and Precalculus

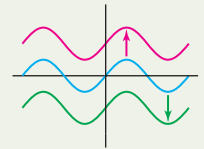
0.1 Functions and Graphs

What Is a Function?
Vertical and Horizontal Line Tests
Properties of Graphs
Examples and Explorations



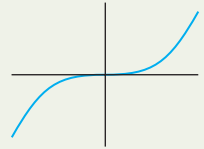
0.2 Operations, Transformations, and Inverses

Combinations of Functions
Transformations and Symmetry
Inverse Functions
Examples and Explorations



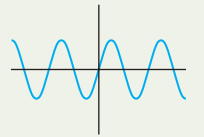
0.3 Algebraic Functions

Power Functions
Polynomial Functions
Rational Functions
Absolute Value Functions
Examples and Explorations



0.4 Exponential and Trigonometric Functions

Exponential Functions
Logarithmic Functions
Trigonometric Functions
Inverse Trigonometric Functions
Examples and Explorations



0.5 Logic and Mathematical Thinking*

From Definitions to Theorems
Quantifiers
Implications
Counterexamples
Simple Mathematical Proofs
Examples and Explorations

$$A \implies B$$

Chapter Review, Self-Test, and Capstones

0.1 FUNCTIONS AND GRAPHS

- ▶ Definition of functions and their domain and range
- ▶ Graphs, horizontal and vertical line tests, and one-to-one-functions
- ▶ Graphical properties and features, asymptotes, and average rate of change

What Is a Function?

Mathematics is a language. In order to understand it, you have to learn how to read it and speak it with the correct vocabulary. Since calculus is at its heart the study of functions of real numbers, the universe we will spend most of our time exploring is the set of real numbers and the relationships between sets of real numbers. Therefore we must begin by setting out the mathematical language that describes these relationships we call “functions.” Once we all speak the same language, we can start building the theory of calculus.

Functions and their properties will be at the core of everything we study in this text. In previous courses you likely encountered functions that were given in terms of formulas, such as

$$y(x) = x^2,$$

that relate two variables x and y . To set the stage for studying such functions, we must first be more precise about what functions are. Instead of thinking of functions merely as formulas, think of them as describing a certain kind of rule, relationship, or mapping from the elements of one set to the elements of another set.

DEFINITION 0.1

Functions

A **function** f from a set A to a set B is an assignment f that associates to each element x of the **domain** set A exactly one element $f(x)$ of the **codomain**, or **target**, set B .

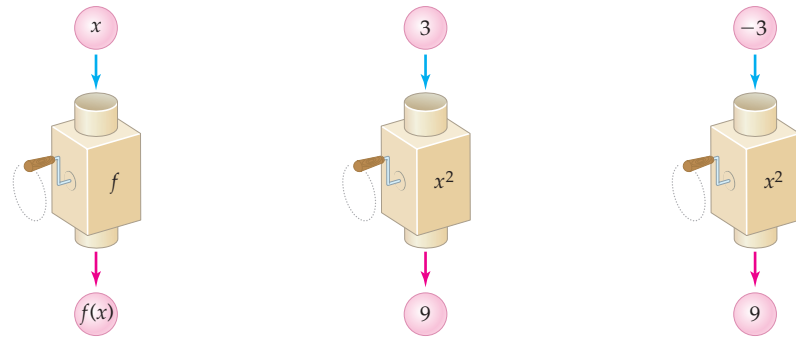
We will use the notation

$$f: A \rightarrow B$$

to represent a function f together with its domain set A and target set B . This notation is pronounced “ f from A to B .” If x and y are variables that represent elements of the sets A and B , respectively, then we say that y is a function of x and write $y = f(x)$ or $y(x)$.

The variable x is called the **independent variable** and represents the “input” of the function. The function f sends each input x to one and only one “output,” some value of the **dependent variable** y . Notice that y depends on x , according to the assignment defined by the function f .

For example, the assignment $f: \mathbb{R} \rightarrow \mathbb{R}$ that squares real numbers is a function, since each real number x is assigned to one and only one real-number square x^2 . Here \mathbb{R} denotes the set of all real numbers, and f assigns each real-number input to exactly one real-number output. Some real numbers (such as 3 and -3) get sent to the same square ($f(-3) = f(3) = 9$), but this does not violate the definition of function. You can think of a function as a machine that takes any given input value x and produces exactly one output value $f(x)$ (pronounced “ f of x ”), shown as follows:



So, what *isn't* a function? If a rule assigns a real-number input to more than one output, then that rule is not a function. For example, consider the formula $y = \pm\sqrt{x}$. This assignment does not define y as a function of x , because the input $x = 4$ corresponds to two different y -values, both $y = -2$ and $y = 2$. In the “function machine” type of illustration just shown, the number 4 would go into the machine and two numbers, -2 and 2 , would come out at once as outputs. This situation is not allowed for functions.

Returning to the squaring function $y = x^2$, notice that some real numbers can never serve as outputs, because squares of real numbers can never be negative. The **range**, or set of possible outputs, of the squaring function is $[0, \infty)$. In this text we will usually work with functions whose domains and ranges are unspecified subsets of real numbers and whose rules are given by formulas such as $f(x) = x^2$.

DEFINITION 0.2 Domain and Range of a Function

If f is a function between unspecified subsets of \mathbb{R} , then we will take the **domain** of f to be the largest subset of \mathbb{R} for which f is defined:

$$\text{Domain}(f) = \{x \in \mathbb{R} \mid f(x) \text{ is defined}\}.$$

The **range** of such a function is the set of all possible outputs that it can attain:

$$\text{Range}(f) = \{y \in \mathbb{R} \mid \text{there is some } x \in \text{Domain}(f) \text{ for which } f(x) = y\}.$$

For example, the function $f(x) = \sqrt{x-1}$ is defined only when $x \geq 1$, and therefore $f(x) = \sqrt{x-1}$ has domain $[1, \infty)$. When we write the square root symbol without the “ \pm ” before it, we always mean the positive square root. This means that $f(x) = \sqrt{x-1}$ can attain only nonnegative y -values. In fact, every nonnegative value can be expressed in the form $\sqrt{x-1}$ for some value of x , and therefore the function $f(x) = \sqrt{x-1}$ has range $[0, \infty)$.

A few notes about the notation we just used: The curly-brackets notation used in Definition 0.2 is called **set notation**, and it is a way to describe a set of real numbers. In this case the set notation for the domain of f is pronounced “the set of all x contained in \mathbb{R} such that $f(x)$ is defined.” Notice in particular that the symbol “ \in ” means “contained in” and the vertical bar means “such that.”

TECHNICAL POINT The name of a function is usually a single letter, such as “ f .” The name of the output of a function f evaluated at an input x is “ $f(x)$.” In this situation f is a function, or relationship, and $f(x)$ is a number that represents the output of the function at the input value x . However, it is sometimes convenient to write $f(x)$ (the name of the output of the function) instead of f (the name of the function itself). This allows us to indicate the name we are using for the independent variable when we reference the function. We may also write things like “consider the function $f(x) = x^2 + 1$,” by which we mean “consider the function f whose output at a real number x is $f(x) = x^2 + 1$.”

Vertical and Horizontal Line Tests

A function whose domain and range are sets of real numbers can be represented as a collection of pairs $(x, f(x))$ of real numbers. If we plot these pairs as points in the Cartesian plane, we obtain the **graph** of the function. In general we have the following definition:

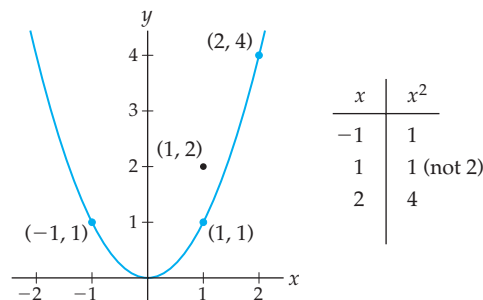
DEFINITION 0.3 The Graph of a Function

The **graph** of a function f is the collection of ordered pairs $(x, f(x))$ for which x is in the domain of f . In set notation we can write

$$\text{Graph}(f) = \{ (x, f(x)) \mid x \in \text{Domain}(f) \}.$$

For example, the graph of $f(x) = x^2$ is the collection of ordered pairs of the form (x, x^2) , for $x \in \mathbb{R}$. Since $f(-1) = (-1)^2 = 1$ and $f(2) = 2^2 = 4$, the points $(-1, 1)$ and $(2, 4)$ are on the graph of $f(x) = x^2$. In contrast, the point $(1, 2)$ is *not* a part of the graph, because $f(1) \neq 2$, as shown in the following graph:

Graph of $f(x) = x^2$ and partial table of values



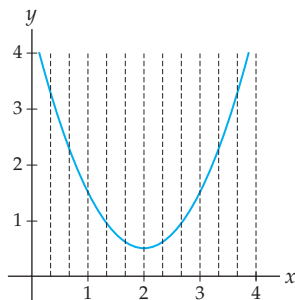
A function always has exactly one output value for every input in the domain, which means that the graph of a function always passes the following test, which you will prove in Exercise 90:

THEOREM 0.4 The Vertical Line Test

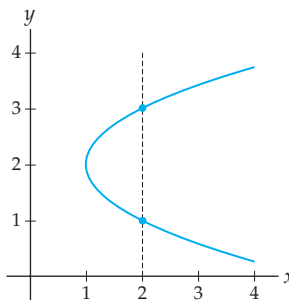
A graph represents a function if and only if every vertical line intersects the graph in at most one point.

For example, consider the three graphs that follow this paragraph. The leftmost graph passes the vertical line test and thus is the graph of a function. The graph in the middle fails the vertical line test because the vertical line $x = 2$ intersects the graph in two points, $(2, 1)$ and $(2, 3)$; therefore the middle graph does not represent a function. The rightmost graph assigns the same output to two distinct inputs, but that is perfectly fine for a function. Because the graph on the right passes the vertical line test, it is the graph of a function.

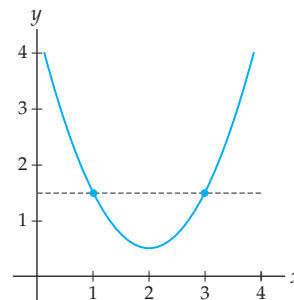
A graph that is a function



A graph that is not a function



A function, but not one-to-one



If each element of the range of a function is the output of exactly one element of the domain, then the function is said to be **one-to-one**. Graphically, we can tell if a function f is one-to-one by checking to see if it passes the **horizontal line test**: if f is one-to-one, then every horizontal line meets the graph of f at most once; see Exercise 91. Algebraically, this means that a function f is one-to-one if two distinct elements in the domain are always sent to different elements of the range:

DEFINITION 0.5**One-to-One Function**

A function f is **one-to-one** if, for all a and b in the domain of f ,

$$a \neq b \implies f(a) \neq f(b).$$

In this definition the notation \implies is pronounced “implies,” and it means that if the left-hand part of the expression is true, then the right-hand part of the expression is also true. In other words, the statement “ $A \implies B$ ” is synonymous with the statement “if A , then B .”

A logically equivalent form of Definition 0.5 is its so-called **contrapositive**:

$$f(a) = f(b) \implies a = b.$$

As we will see in Section 0.5, the contrapositive of an implication $A \implies B$ is the equivalent statement $(\text{not } B) \implies (\text{not } A)$. The contrapositive form of Definition 0.5 is often easier to use, because it is an affirmative rather than a negative statement. For example, $f(x) = 3x$ is one-to-one because if $3a = 3b$, then we can guarantee that $a = b$. In contrast, the squaring function $f(x) = x^2$ is not one-to-one, because we cannot guarantee that if $a^2 = b^2$, then $a = b$ (for example, $(-3)^2 = 3^2$, but $-3 \neq 3$).

Properties of Graphs

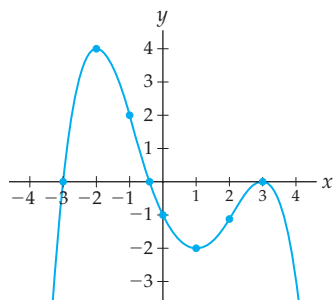
The table that follows gives us vocabulary and precise mathematical definitions for various types of graphical behavior. Rows 1, 2, 5, and 6 describe behaviors that a function could exhibit at a specific point. The remaining rows describe graphical behaviors that occur over an interval I of real numbers. Much of the material in the early chapters of this book will be dedicated to developing techniques for properly defining and identifying these types of

properties of functions. For now we present them just to set terminology and to familiarize ourselves with various types of function behavior.

Vocabulary	Definition	Behavior
f has a root at $x = c$	$f(c) = 0$	graph intersects the x -axis at $x = c$
f has a y-intercept at $y = b$	$f(0) = b$	graph intersects the y -axis at $y = b$
f is positive on I	$f(x) > 0$ for all $x \in I$	graph is above the x -axis on I
f is increasing on I	$f(b) > f(a)$ for all $b > a$ in I	graph moves up as we look from left to right on I
f has a local maximum at $x = c$	$f(c) \geq f(x)$ for all x near $x = c$	graph has a relative “hilltop” at $x = c$
f has a global maximum at $x = c$	$f(c) \geq f(x)$ for all $x \in \text{Domain}(f)$	graph is the highest at $x = c$
f is concave up on I	<i>will state precisely in Section 3.3</i>	graph curves upwards on I like part of a “U”
f has an inflection point at $x = c$	<i>will state precisely in Section 3.3</i>	graph of f changes concavity at $x = c$

Of course, there are similar definitions for local and global minima and for negative, decreasing, and concave-down behavior; see Exercises 20 and 21. Notice that we describe **extrema** (maxima and minima) by *where* on the x -axis they occur, since we can always find the corresponding y -values from these x -values. The concept of “near” in the description of a local maximum will be made more precise in Chapters 1 and 2. Inflection points and concavity cannot be precisely defined until we learn about **derivatives** in Chapters 2 and 3. In that chapter we will also learn ways for algebraically calculating the locations of extrema and inflection points. Until then, we will have to be content with examining such things graphically.

For example, the list that follows at the right describes some aspects of the graphical behavior of the graph $y = f(x)$ shown on the left.



- ▶ roots at $x = -3$, $x \approx -0.4$, and $x = 3$
- ▶ y -intercept at $y = -1$
- ▶ local maxima at $x = -2$ and $x = 3$
- ▶ global maximum at $x = -2$
- ▶ inflection points at $x = -1$ and $x = 2$
- ▶ positive on $(-3, -0.4)$
- ▶ increasing on $(-\infty, -2)$ and $(1, 3)$
- ▶ concave up on $(-1, 2)$

In fact, technically the function f graphed at the left is increasing on the larger intervals $(-\infty, -2]$ and $[1, 3]$. This is because we do have $f(b) > f(a)$ for all values $b > a$ in these *closed* intervals. Most of the time we will be concerned only with the open intervals on which

a function is increasing or decreasing, but there will be a few times in later chapters when we need to consider closed or half-closed intervals. For now, we will use open intervals unless we require otherwise.

The increasing and/or decreasing behavior of a function is related to its **average rate of change** on various intervals. The average rate of change of a function f on an interval $[a, b]$ measures how much the output $f(x)$ changes over that interval. Average rates of change will be extremely important in Chapter 2 when we study the derivative.

DEFINITION 0.6

Average Rate of Change

The **average rate of change** of a function f on an interval $[a, b]$ is the slope of the line from $(a, f(a))$ to $(b, f(b))$, which is given by the quotient

$$\frac{f(b) - f(a)}{b - a}.$$

For example, the function whose properties we just enumerated is increasing on the interval $(1, 3)$, moving up from $(1, f(1)) = (1, -2)$ to $(3, f(3)) = (3, 0)$. The average rate of change tells us how much the function increased per unit change in the input, on average:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{0 - (-2)}{2} = 1$$

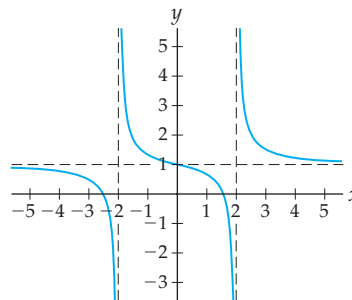
unit up for every unit across. We can also measure average rate of change over intervals where the function both increases and decreases; for example, with the same function, on the interval $[-3, 3]$ there is an average rate of change of

$$\frac{f(3) - f(-3)}{3 - (-3)} = \frac{0 - 0}{6} = 0$$

units up for every unit across; look at the graph to see why this makes sense.

Sometimes a graph gets closer and closer to a horizontal or vertical line, or **asymptote**. In Chapter 1, we will define asymptotes precisely, using limits. For now, we will use the following definition: A line l is an **asymptote** of a function f if the difference between the graph of l and the graph of f gets as small as we want as either x or y increases in magnitude. For example, the following graph of a function f has vertical asymptotes at $x = -2$ and $x = 2$, and a horizontal asymptote at $y = 1$:

A function with three asymptotes



Notice that a graph *can* cross one of its horizontal asymptotes; the preceding graph above does so at the point $(0, 1)$. This is just one of the reasons that we are avoiding using the overly loose definition of asymptote that you may have heard in previous courses (“an asymptote is a line that the graph gets infinitely close to, but never reaches”).

Examples and Explorations

EXAMPLE 1

Identifying functions and their domains and ranges

Determine whether or not each of the following relationships is a function. For each relationship that is a function, describe its natural domain and range and determine whether or not it is one-to-one. For each relationship that is not a function, describe the parts of the definition of a function that are violated.

(a) The rule $g: \mathbb{R} \rightarrow \mathbb{R}$ that assigns each real number x to the numbers whose square is x .

(b) The relationship defined by this table:

x	1	2	3	4	5	6
$P(x)$	5	2	9	-1	0	9

(c) Let P be the set of all living people in the world, and let W be the set of all women that have ever lived. Define $f: P \rightarrow W$ so that each person is assigned to his or her biological mother.

(d) $f(x) = 2 - \sqrt{x+5}$

(e) $h(x) = \frac{1}{x^2 - 4}$

SOLUTION

(a) This rule is not a function, for two reasons. First of all, negative real numbers do not have real square roots, so g is not defined on the given domain of \mathbb{R} . Second, each positive number x has two numbers whose square is x , namely, \sqrt{x} and $-\sqrt{x}$, so this rule would not send each domain element to exactly one output.

(b) The relationship $P(x)$ defined by the table is a function, because the table assigns each value in the domain $\{1, 2, 3, 4, 5, 6\}$ to exactly one element of the range $\{-1, 0, 2, 5, 9\}$. This function is not one-to-one because $P(3)$ and $P(6)$ are both equal to 9.

(c) This relationship is a function because each person has one exactly one woman who is his or her biological mother. No person is without a biological mother, and no person has more than one biological mother. Here the domain is P and the range is the subset of W consisting of women that have had biological children. This function is not one-to-one, since there are examples of different people that have the same biological mother.

(d) This rule is a function because for each value x for which the formula makes sense, there is exactly one real number described by $2 - \sqrt{x+5}$. For x to be in the domain, we must have $x+5 \geq 0$ (since $x+5$ is under a square-root sign), and thus we must have $x \geq -5$. Therefore the domain of f is $[-5, \infty)$. The range of $y = f(x)$ is the set of y -values that can occur as outputs. Since $\sqrt{x+5}$ can take on any value greater than or equal to 0, the expression $2 - \sqrt{x+5}$ can take on any value less than or equal to 2. Therefore the range of f is $(-\infty, 2]$. This function is one-to-one because if $f(a) = f(b)$, then

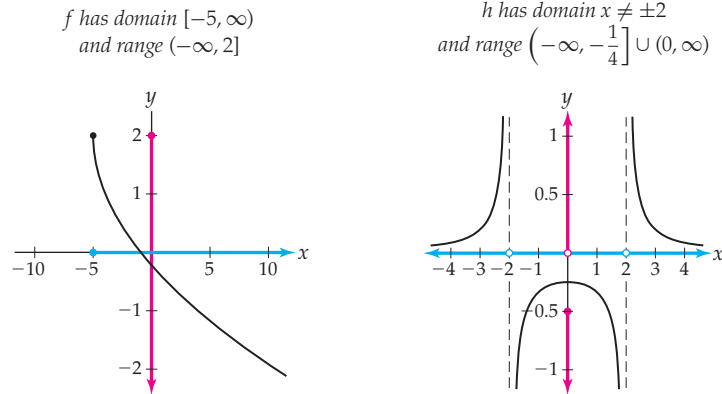
$$\begin{aligned} 2 - \sqrt{a+5} &= 2 - \sqrt{b+5} &\implies \sqrt{a+5} &= \sqrt{b+5} \\ & &\implies a+5 &= b+5 \\ & &\implies a &= b. \end{aligned}$$

(e) The rule $h(x)$ is a function because for each value x at which $\frac{1}{x^2-4}$ is defined, there is exactly one real number that h describes. The domain of $h(x)$ is the set of all x -values for which $x^2 - 4 \neq 0$ (since $x^2 - 4$ is in a denominator). Therefore the domain of h is everything except $x = \pm 2$. To find the range of $h(x)$ we must find the y -values that can be expressed in the form $y = h(x)$ for some x . Solving for x in terms of y we obtain $x = \sqrt{\frac{1}{y} + 4}$. This means we can find an x that maps via f to y as long as $y \neq 0$ and $\frac{1}{y} + 4 \geq 0$. It can be shown that the solution of the latter inequality is

$(-\infty, -\frac{1}{4}] \cup [0, \infty)$. Therefore the range of $h(x)$ is $(-\infty, -\frac{1}{4}] \cup (0, \infty)$. This function is not one-to-one because, for example, $h(1)$ and $h(-1)$ are both equal to $-\frac{1}{3}$. \square

CHECKING THE ANSWER

The following functions f and h have domains marked in blue on the x -axis and ranges marked in red on the y -axis:



EXAMPLE 2 Evaluating function notation

Given that $f(x) = x\sqrt{3-x}$, evaluate $f(2)$, $f(a)$, $f(x+1)$, and $f(f(x))$.

SOLUTION

To evaluate $f(x) = x\sqrt{3-x}$ at a given input, simply replace x in the formula with whatever the input is:

$$\begin{aligned} f(2) &= 2\sqrt{3-2} = 2\sqrt{1} = 2; \\ f(a) &= a\sqrt{3-a}; \\ f(x+1) &= (x+1)\sqrt{3-(x+1)}; \\ f(f(x)) &= f(x)\sqrt{3-f(x)} = (x\sqrt{3-x})\sqrt{3-x\sqrt{3-x}}. \end{aligned} \quad \square$$

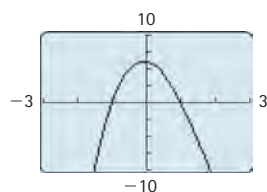
EXAMPLE 3 Finding a “good” graphing window

Use a graphing utility to find a graphing window that accurately represents the key features of the graph of the function $f(x) = x^3 - 6x^2 - x + 6$.

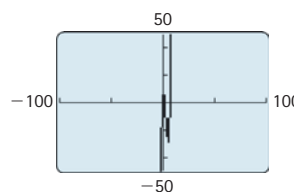
SOLUTION

The three graphs that follow show $y = f(x)$ in various graphing windows. Each of these windows is “bad” in the sense that the true behavior of the graph of f is not represented.

$$f(x) = x^3 - 6x^2 - x + 6 \\ x \in [-3, 3], y \in [-10, 10]$$



$$f(x) = x^3 - 6x^2 - x + 6 \\ x \in [-100, 100], y \in [-50, 50]$$



$$f(x) = x^3 - 6x^2 - x + 6 \\ x \in [-20, 20], y \in [-1000, 1000]$$

